ISRAEL JOURNAL OF MATHEMATICS, Vol. 54, No. 2, 1986

BI-CONVEXITY AND BI-MARTINGALES†

BY

ROBERT J. AUMANN[a] AND SERGIU HART[b]

[a]*Department of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel;*
and [b]*School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel*

ABSTRACT

A set in a product space $\mathscr{X} \times \mathscr{Y}$ is *bi-convex* if all its x- and y-sections are convex. A *bi-martingale* is a martingale with values in $\mathscr{X} \times \mathscr{Y}$ whose x- and y-coordinates change only one at a time. This paper investigates the limiting behavior of bimartingales in terms of the *bi-convex hull* of a set — the smallest bi-convex set containing it — and of several related concepts generalizing the concept of separation to the bi-convex case.

0. Introduction

Let $\mathscr{X}$ and $\mathscr{Y}$ be compact convex subsets of Euclidean spaces (usually of different dimensions), with generic elements x and y. A subset of $\mathscr{X} \times \mathscr{Y}$ is *bi-convex* if each of its x- and y-sections is convex. The *bi-convex hull* of a set is the smallest bi-convex set containing it. A real function $f(x, y)$ on a bi-convex subset of $\mathscr{X} \times \mathscr{Y}$ is *bi-convex* if it is convex in each variable x and y separately. A *bi-martingale* is a martingale with values in $\mathscr{X} \times \mathscr{Y}$ whose x- and y-coordinates change only one at a time. (For detailed definitions and illustrative examples, see Sections 1, 2 and 3.)

These concepts arise in the analysis of repeated games of incomplete information [3]. In this paper we explore the relationships between them.

A martingale can be viewed as a splitting process. A particle (mass point) in space splits into several new particles, whose centroid is the starting point. Each of the new particles then splits, and the process is repeated again and again.

† Research partially supported by NSF grants at the Institute for Mathematical Studies in the Social Sciences, Stanford University. The second author has also been partially supported by the Deutche Forschungsgemeinschaft. We thank Andreu Mas-Colell, Jean-Francois Mertens, Abraham Neyman and Lloyd S. Shapley for many useful discussions.

Received November 15, 1983 and in revised form June 15, 1985

Eventually a cloud forms; if we confine ourselves to a bounded subset of space, then by the martingale convergence theorem, the cloud converges to a limit cloud. At each stage, the starting point is the centroid of the cloud, and therefore lies in its convex hull. It also lies in the convex hull of the limit cloud. (Here, mass corresponds to probability, and centroid to expectation.)

If the martingale is a bi-martingale, at each stage either all particles split "horizontally", or all particles split "vertically". Therefore, at each stage the starting point is in the bi-convex hull of the cloud. Rather surprisingly, though, it need not be in the bi-convex hull of the limit cloud (see Example 2.5). This is so even in the special case in which the bi-martingale is *almost finite* (i.e. the mass that continues to split after n stages tends to 0 as $n \to \infty$).

Given the limit cloud, what *can* we say about the starting point? To answer this question, we must examine more carefully the notion of convex hull and its generalizations to bi-convexity. The convex hull $\mathrm{co}(A)$ of a set A can be defined as the smallest convex set containing A; this is the definition that corresponds to the definition of bi-convex hull given above. But $\mathrm{co}(A)$ can also be defined by a process of separation, as follows: First, one removes all the points z that can be strictly separated from A by a convex function f (i.e., $f(z) > \sup f(A)$). This yields the closed convex hull B_1 of A; obviously $B_1 \supset A$. Define B_2 by removing from B_1 all points that can be strictly separated from A by a convex function defined on B_1 only. The reader may convince himself that iterated finitely often, this process leads to $\mathrm{co}(A)$, where it ends.

One may also apply this process of separation to bi-convexity, substituting bi-convex functions for convex functions. The process may then require transfinitely many iterations; but it, too, must eventually end. We call the result $\text{bi-co}^{\#}(A)$; it always contains $\text{bi-co}(A)$, but, unlike in the case of convexity, it is in general different (Example 2.5).

Suppose that in the iterative process that leads to $\mathrm{co}(A)$, we limit ourselves to separating functions that, in addition to being convex, are continuous. Then it may be seen that we will never get beyond the closed convex hull — the first iteration will also be the last. But if we demand that the separating functions be continuous only on A, then again, a finite number of iterations lead to $\mathrm{co}(A)$.

Similarly, in the case of bi-convexity we may separate by bi-convex functions that are continuous on A. Again, the process must converge (after a possibly transfinite number of stages). The result, which we call $\text{bi-co}^*(A)$, may be different both from $\text{bi-co}(A)$ and from $\text{bi-co}^{\#}(A)$ (see Section 5); of course, $\text{bi-co}(A) \subset \text{bi-co}^{\#}(A) \subset \text{bi-co}^*(A)$.

Our main results (Section 4) may now be stated as follows:

(1) If A is the limit cloud of a bi-martingale, then the set of all possible starting points is bi-co*(A) (see Theorem 4.7).

(2) If we restrict ourselves to bi-martingales that are almost finite (see definition above), then the set of all possible starting points is bi-co$^{\#}(A)$ (Theorem 4.3).

To complete the picture, we note that

(3) If we restrict ourselves to *finite* bi-martingales (i.e. those that actually remain fixed after a bounded number of stages), then the set of all possible starting points is bi-co(A).

1. Bi-martingales

Let $\mathscr{X}$ and $\mathscr{Y}$ be compact convex subsets of some Euclidean spaces (of different dimensions, in general). Let $(\Omega, \mathscr{F}, P)$ be an atomless probability space. A sequence $\{Z_n\}_{n=1}^{\infty} \equiv \{(X_n, Y_n)\}_{n=1}^{\infty}$ of $(\mathscr{X} \times \mathscr{Y})$-valued random variables is a *bi-martingale* if:

(1.1) There exists a non-decreasing sequence $\{\mathscr{F}_n\}_{n=1}^{\infty}$ of finite subfields[1] of $\mathscr{F}$, such that $\{Z_n\}_n$ is a martingale with respect to $\{\mathscr{F}_n\}_n$.

(1.2) For each $n = 1, 2, \ldots$, either $X_n = X_{n+1}$ or $Y_n = Y_{n+1}$ (a.s.).

(1.3) Z_1 is constant (a.s.).

The martingale condition (1.1) means, first, that Z_n is $\mathscr{F}_n$-measurable, and second, that $E(Z_{n+1} \mid \mathscr{F}_n) = Z_n$ (a.s.), for all $n = 1, 2, \ldots$. By (1.3), we thus have $E(Z_n) = Z_1$ for all n. Since $\mathscr{X}$ and $\mathscr{Y}$ are compact, the sequence $\{Z_n\}$ forms a bounded martingale, hence it has an almost everywhere limit $Z_\infty \equiv (X_\infty, Y_\infty)$.

Let A now be a measurable subset of $\mathscr{X} \times \mathscr{Y}$. We will consider the following set:

(1.4) $$A^* = \{z \in \mathscr{X} \times \mathscr{Y} \mid \text{there exists a bi-martingale } \{Z_n\}_{n=1}^{\infty} \text{ converging to } Z_\infty, \text{ such that } Z_\infty \in A \text{ and } Z_1 = z \text{ (a.s.)}\}.$$

Without condition (1.2), A^* becomes just co(A), the convex hull[2] of A; the

[1] A field is finite if it contains finitely many elements; this finiteness condition will turn out to be inessential — see Remark 4.11.

[2] Indeed, every point in co(A) can be obtained (by Caratheodory's theorem). Conversely, we have $z = E(Z_\infty)$ where $P(Z_\infty \in A) = 1$, which implies $z \in \overline{\text{co}}(A)$ (= the closed convex hull of A). If $z \notin \text{co}(A)$, then there exists a supporting hyperplane, i.e., $\lambda \neq 0$ such that $\lambda \cdot z = \sup\{\lambda \cdot a \mid a \in A\}$. But this implies $P(Z_\infty \in A') = 1$, where $A' = \{a \in A \mid \lambda \cdot a = \lambda \cdot z\}$, and A' is a set of lower dimension than A. The proof is now completed by induction.

same will happen if we drop (1.3) (and replace $Z_1 = z$ by $E(Z_1) = z$ in the definition of A^*). However, the set A^* as given by (1.4) is in general strictly included in $\mathrm{co}(A)$. For example, as we will see later, if we take $\mathscr{X} = \mathscr{Y} = [0,1]$ and $A = \{(0,0),(1,0),(0,1)\}$, then A^* is the L-shaped set $\{(x,y) \in [0,1] \times [0,1] \mid x = 0 \text{ or } y = 0\}$.

REMARK 1.5. One can represent a bi-martingale $\{Z_n\}_{n=1}^{\infty}$ as a rooted tree, with the values of Z_n attached to its nodes at level n, and the probabilities $P(Z_{n+1} \mid \mathscr{F}_n)$ attached to its branches from there. For example, see Fig. 1.1, where

$$1 = \alpha_1 + \beta_1 = \alpha_2^1 + \beta_2^1 = \alpha_2^2 + \beta_2^2 = \dots; \qquad 0 \leqq \alpha_1, \beta_1, \alpha_2^1, \beta_2^1, \alpha_2^2, \beta_2^2, \dots;$$

$$z_1 = \alpha_1 z_2^1 + \beta_1 z_2^2, \qquad z_2^1 = \alpha_2^1 z_3^1 + \beta_2^1 z_3^2, \ \dots;$$

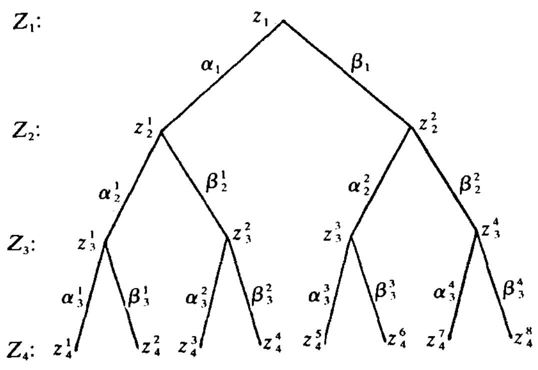

Fig. 1.1.

and (writing $z_i^j \equiv (x_i^j, y_i^j)$)

$$x_1 = x_2^1 = x_2^2, \quad y_2^1 = y_3^1 = y_3^2, \quad y_2^2 = y_3^3 = y_3^4, \ \dots.$$

Note that the total probability of each node is the product of the probabilities along the unique path connecting the node to the root.

Conversely, each such tree structure gives rise to a bi-martingale; this is the (only) reason we required P to be an atomless measure. It follows that the specific choice of the probability space is of no consequence, as long as it is atomless. Thus, A^* is determined by *distributions* of bi-martingales, and not by the bi-martingales themselves.

REMARK 1.6. If $\{Z_n\}_{n=1}^{\infty}$ is a bi-martingale, $Z_n \to Z_\infty$ and $Z_n \in A$ a.s., then $Z_n \in A^*$ for all n. But $A^* \subset \mathrm{co}(A)$, therefore A^* does not change if we replace $\mathscr{X}$ and $\mathscr{Y}$ by other compact convex sets whose product contains A.

REMARK 1.7. Call a bi-martingale *binary* if all nodes in the associated tree (cf. Remark 1.5) have at most two immediate successors. Note that A^* does not change if we consider only binary bi-martingales. The interest in these is due to the following: Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of $(\mathscr{X} \times \mathscr{Y})$-valued random variables, where $Z_n \equiv (X_n, Y_n)$, and denote by $X_n^{(i)}$ and $Y_n^{(i)}$ the coordinates of X_n and Y_n, respectively. Then $\{Z_n\}$ is a binary bi-martingale if and only if it satisfies (1.1), (1.3), and for each i and j, the sequence $\{X_n^{(i)} \cdot Y_n^{(j)}\}_{n=1}^{\infty}$ is a real-valued martingale (with respect to the same $\{\mathscr{F}_n\}_{n=1}^{\infty}$ as in (1.1)). This follows from the easily checked fact that for real numbers, if $x = \alpha x' + (1-\alpha)x''$, $y = \alpha y' + (1-\alpha)y''$, $xy = \alpha x'y' + (1-\alpha)x''y''$ and $0 < \alpha < 1$, then either $x = x' = x''$ or $y = y' = y''$. It is however no longer true for convex combinations of more than two points; e.g.,

$$(\tfrac{3}{2}, \tfrac{3}{2}; \tfrac{3}{2} \cdot \tfrac{3}{2}) = \tfrac{1}{4}(0,0;0 \cdot 0) + \tfrac{3}{8}(3,1;3 \cdot 1) + \tfrac{3}{8}(1,3;1 \cdot 3).$$

2. Bi-convex sets

A convex combination $(x, y) = \sum_{i=1}^{m} \alpha_i (x_i, y_i)$ (with $\alpha_i \geqq 0$, $\sum_{i=1}^{m} \alpha_i = 1$) will be called *bi-convex* if either $x_1 = x_2 = \cdots = x_m = x$ or $y_1 = y_2 = \cdots = y_m = y$. A set B is a *bi-convex set* if it contains all the bi-convex combinations of its elements. Thus, B is bi-convex if for all $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, its sections $B_{x\cdot} \equiv \{y \in \mathscr{Y} \mid (x, y) \in B\}$ and $B_{\cdot y} \equiv \{x \in \mathscr{X} \mid (x, y) \in B\}$ are convex sets. An example of a bi-convex set that is not convex is again $B = \{(x, y) \in [0,1] \times [0,1] \mid x = 0$ or $y = 0\}$. Another example is the graph of the subdifferential mapping of a convex function (cf. [5], Theorem 23.5).

Next, we want to define the bi-convex hull of a given subset A of $\mathscr{X} \times \mathscr{Y}$. There are two ways to proceed.

First, define inductively the sequence of sets $\{A_n\}_{n=1}^{\infty}$ as follows: $A_1 = A$ and A_{n+1} is the set of all bi-convex combinations of elements of A_n (for $n = 1, 2, \ldots$). Let $B = \bigcup_{n=1}^{\infty} A_n$ be the limit of this sequence. Second, let B' be the intersection of all bi-convex sets that contain A.

PROPOSITION 2.1. *$B = B'$ = the smallest[3] bi-convex set containing A.*

The proof is straightforward; we will call the set obtained the *bi-convex hull* of A, and will denote it $\text{bi-co}(A)$.

[3] Relative to set inclusion.

An interesting question is: does there exist an n such that, analogous to Caratheodory's theorem, $\text{bi-co}(A) = A_n$? The answer is (in general) no.

EXAMPLE 2.2. Let $\mathscr{X} = \mathscr{Y} = [0,1]$, and for all $m = 1,2,\dots$ define

$$z_{2m} = \left(1 - \frac{1}{2^{m-1}}, 1 - \frac{3}{2^{m+2}}\right), \qquad z_{2m+1} = \left(1 - \frac{3}{2^{m+2}}, 1 - \frac{1}{2^m}\right),$$

$$w_{2m} = \left(1 - \frac{1}{2^{m-1}}, 1 - \frac{1}{2^m}\right), \qquad w_{2m+1} = \left(1 - \frac{1}{2^m}, 1 - \frac{1}{2^m}\right),$$

and put $z_1 = w_1 = (0,0)$. Then w_n is a bi-convex combination of z_n and w_{n-1} (for $n = 2,3,\dots$), namely $w_n = \frac{4}{5} z_n + \frac{1}{5} w_{n-1}$. Now let $A = \{z_n\}_{n=1}^{\infty}$; then it can be checked that $w_n \in A_n$ but $w_n \not\in A_{n-1}$, for each $n = 2,3,\dots$ (see Fig. 2.1). ■

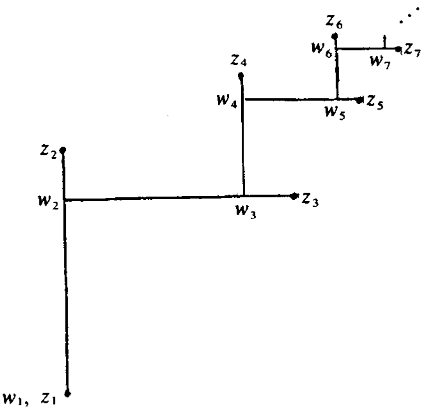

Fig. 2.1.

Note that by adding the point $(1,1)$ to the set A in Example 2.2, one obtains a *closed* (hence compact) set A with $\text{bi-co}(A) \supsetneqq A_n$ for all n.

How are bi-convex sets related to bi-martingales?

PROPOSITION 2.3. *For any set A, A^* is a bi-convex set containing* $\text{bi-co}(A)$.

PROOF. To see that A^* is a bi-convex set, recall the tree structure in Remark 1.5. Given a collection of m such trees, with roots $z_1, \dots, z_m$, where, say, $x_1 = \cdots = x_m = x$, we construct for every non-negative $\alpha_1, \cdots, \alpha_m$ with $\sum_{i=1}^m \alpha_i = 1$ a new tree as follows. The root is $z = (x, y)$, where $y = \sum_{i=1}^m \alpha_i y_i$; it has m branches to nodes $z_1, \dots, z_m$, with probabilities $\alpha_1, \dots, \alpha_m$ (respectively); from each such node z_i, we follow the corresponding given tree. This shows that if $z_1, \dots, z_m$ belong to A^*, then $z \in A^*$ too.

The inclusion $A^* \supset A$ is obtained by considering constant bi-martingales; it implies that $A^* \supset \text{bi-co}(A)$. ∎

REMARK 2.4. The set A_n corresponds precisely to those bi-martingales $\{Z_m\}_{m=1}^{\infty}$ for which the limit Z_∞ is attained at most in n steps (i.e., $Z_n = Z_\infty$).

Are the two sets A^* and $\text{bi-co}(A)$ actually equal? The following example shows that this is not the case in general.

EXAMPLE 2.5. Again, let $\mathscr{X} = \mathscr{Y} = [0,1]$. Let $z_1 = (\frac{1}{3}, 0)$, $z_2 = (0, \frac{2}{3})$, $z_3 = (\frac{2}{3}, 1)$ and $z_4 = (1, \frac{1}{3})$, then $A = \{z_1, z_2, z_3, z_4\}$ is clearly a bi-convex set, i.e., $A = \text{bi-co}(A)$. Let $w_1 = (\frac{1}{3}, \frac{1}{3})$, $w_2 = (\frac{1}{3}, \frac{2}{3})$, $w_3 = (\frac{2}{3}, \frac{2}{3})$ and $w_4 = (\frac{2}{3}, \frac{1}{3})$; we will show that all these points belong to A^* (as we will see in Section 4, A^* is precisely the bi-convex hull of all the points z_i and w_i, $1 \leqq i \leqq 4$; it consists of the square whose vertices are the w_i's, together with the four line segments $[w_i, z_i]$; see Fig. 2.2).

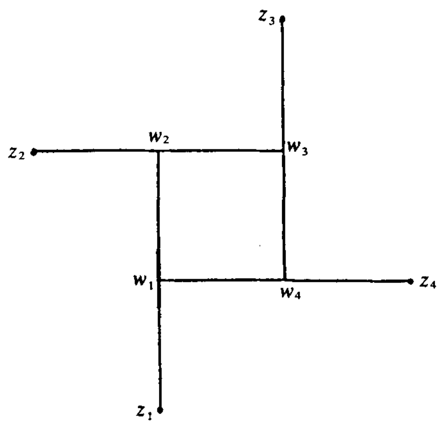

Fig. 2.2.

Indeed, consider the following tree (see Fig. 2.3): the root is w_1; every node w_i has two sons, z_i and w_{i+1} (where $i+1$ is taken modulo 4), with probability $\frac{1}{2}$ each; every node z_i has one son z_i only. It is easily seen that this tree defines a bi-martingale $\{Z_n\}_{n=1}^{\infty}$ with $Z_1 = w_1$; the probability that A is never reached is zero (this happens only along the rightmost path in the tree, whose probability is $\lim_{n\to\infty}(\frac{1}{2})^n = 0$), thus $w_1 \in A^*$. A similar construction proves that w_2, w_3 and w_4 belong to A^* too. ∎

This example points out the difference between "finite" bi-martingales (which generate only $\bigcup_{n=1}^{\infty} A_n = \text{bi-co}(A)$; see Remark 2.4), and "infinite" ones. In Section 4 we will make this distinction (and another one) more precise.

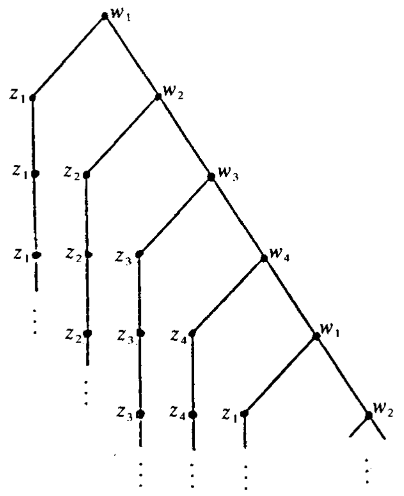

Fig. 2.3.

3. Bi-convex functions

In the previous section we saw that bi-convex sets are not sufficient to characterize A^*. We thus approach the problem in a dual way — by separation. In the case of convexity, it is enough to consider affine[4] functions: any point outside a convex set can be separated from it by such a function. However, this does not generalize to the bi-convex case: the corresponding bi-affine functions separate strictly less than the larger class of bi-convex functions.

Let $B \subset \mathscr{X} \times \mathscr{Y}$ be a bi-convex set, and let $f: B \to \mathbf{R}$, where $\mathbf{R}$ denotes the real line. The function f is *bi-convex* (*bi-affine*) if $f(x, \cdot)$ is a convex (affine) function on $B_{x\cdot} = \{y \in \mathscr{Y} \mid (x, y) \in B\}$ for all $x \in \mathscr{X}$, and $f(\cdot, y)$ is a convex (affine) function on $B_{\cdot y} = \{x \in \mathscr{X} \mid (x, y) \in B\}$ for all $y \in \mathscr{Y}$; i.e.,

$$f(\lambda' x' + \lambda'' x'', y) \leqq \lambda' f(x', y) + \lambda'' f(x'', y)$$

and

$$f(x, \lambda' y' + \lambda'' y'') \leqq \lambda' f(x, y') + \lambda'' f(x, y'')$$

for all $\lambda', \lambda'' \geqq 0$, $\lambda' + \lambda'' = 1$, and $(x', y), (x'', y), (x, y'), (x, y'') \in B$. Note that f is bi-affine if we have equalities above; it has to be of the form

$$f(x, y) = \sum_{i,j} \alpha_{ij} x^{(i)} y^{(j)} + \sum_i \beta_i x^{(i)} + \sum_j \gamma_j y^{(j)} + \delta,$$

[4] We use *affine* for a function that is both convex and concave; it is sometimes called *linear*.

where i and j denote the coordinates of x and y, respectively, and α_{ij}, β_i, γ_j and δ are real constants.

The following is immediate.

PROPOSITION 3.1. *Let $f: B \to \mathbf{R}$ be a bi-convex function.*[5] *Then, for all real α, the set $\{(x, y) \in B \mid f(x, y) \leqq \alpha\}$ is a bi-convex set.*

As in the standard convex case, the converse is of course not true in general.

We can now define the notion of separation. Let B be a bi-convex set, $B \supset A$ (the set A is assumed fixed throughout). Then a point $z \in B$ is (*strongly bi-*) *separated from A with respect to*[6] B if there exists a bounded bi-convex function f on B such that $f(z) > \sup f(A) \equiv \sup\{f(a) \mid a \in A\}$. Let us denote by $\mathrm{ns}(B)(\equiv \mathrm{ns}_A(B))$ the set of all points $z \in B$ that cannot be separated from A; thus, $z \in \mathrm{ns}(B)$ if and only if $z \in B$ and, for all bi-convex functions f defined on B, we have $f(z) \leqq \sup f(A)$.

Form Proposition 3.1 one readily obtains

PROPOSITION 3.2. *Let B be a bi-convex set, $B \supset A$. Then the set* $\mathrm{ns}(B)$ *is bi-convex, and* $\mathrm{ns}(B) \supset \text{bi-co}(A)$.

In general, we cannot expect the opposite inclusion to hold, since even for ordinary convexity, the analogous assertion cannot be made: if B is a convex set and $A \subset B$, then the set of points in B that cannot be separated from A by a convex function on B need *not* be included in $\mathrm{co}(A)$. This set *is*, however, included in $\overline{\mathrm{co}}(A)$, the closed convex hull of A, and so the question arises whether, similarly, we can assert that $\mathrm{ns}(B)$ is included in $\overline{\text{bi-co}}(A)$. The answer is no; this is further evidence for the non-finite dimensional character of bi-convexity (see Example 2.2).

EXAMPLE 3.3. Consider again Example 2.5, and let $B = \mathscr{X} \times \mathscr{Y}$. Let f be a bi-convex function on B, and assume that it separates at least one of the points w_1, w_2, w_3, w_4 from A. Let i be such that $f(w_i) \geqq f(w_j)$ for all $1 \leqq j \leqq 4$, then f separates w_i from A. Now f is bi-convex, thus

$$f(w_i) \leqq \tfrac{1}{2} f(z_i) + \tfrac{1}{2} f(w_{i-1})$$

(where we define $w_0 \equiv w_4$). But $z_i \in A$, thus $f(w_i) > f(z_i)$, which implies $f(w_{i-1}) > f(w_i)$, contradicting the choice of i. This shows that $w_i \in \mathrm{ns}(B)$ for $1 \leqq i \leqq 4$. On the other hand, $w_i \notin A$; since A is itself closed and bi-convex, it

[5] We always assume that the domain of definition B of a bi-convex function is a bi-convex set.

[6] The domain does indeed matter — see Example 3.5.

follows that $\mathrm{ns}(B)$ is *not* included in $\overline{\text{bi-co}}(A)$. From $w_i \in \text{bi-co}(A)$ it follows that $\mathrm{ns}(B) \supset C \equiv \text{bi-co}\{z_i, w_i \mid 1 \leqq i \leqq 4\}$ (by Proposition 3.2). At the end of this section we will show that actually $\mathrm{ns}(B) = C$. ■

We claimed that separation by bi-affine functions is not sufficient; the following example shows that bi-convex functions may indeed separate more.

EXAMPLE 3.4. Let $\mathscr{X} = \mathscr{Y} = [0,1]$, $A = \{(0,0), (\frac{1}{2},0), (0,\frac{1}{2}), (1,1)\}$, $B = \mathscr{X} \times \mathscr{Y}$ (see Fig. 3.1). It is easy to see that $\text{bi-co}(A)$ consists of A together with the two line segments $[(0,0),(\frac{1}{2},0)]$ and $[(0,0),(0,\frac{1}{2})]$. Consider now the following function f on B:

$$f(x,y) = \begin{cases} xy, & 0 \leqq x, y \leqq 1/2, \\ -3xy + 2x + 2y - 1, & \text{otherwise.} \end{cases}$$

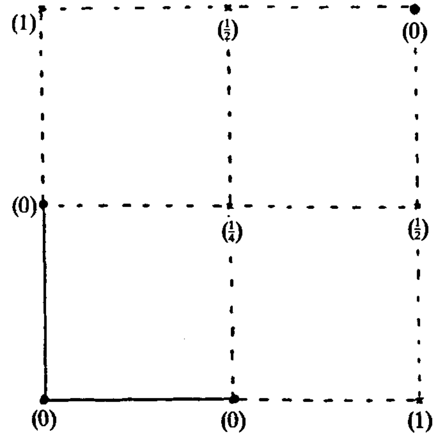

Fig. 3.1.

It can be checked that $f(x,y) \geqq 0$ for all $(x,y) \in [0,1] \times [0,1]$, $f(x,y) = 0$ if and only if $(x,y) \in \text{bi-co}(A)$, and f is bi-convex (actually, it is *piecewise bi-affine*; it is obtained by putting $f(0,0) = f(0,\frac{1}{2}) = f(\frac{1}{2},0) = f(1,1) = 0$, $f(\frac{1}{2},\frac{1}{2}) = \frac{1}{4}$, $f(0,1) = f(1,0) = 1$ and $f(\frac{1}{2},1) = f(1,\frac{1}{2}) = \frac{1}{2}$, and then extending it bi-affinely in each of the four small squares). Therefore, f separates every point not in $\text{bi-co}(A)$ from A; thus, $\mathrm{ns}(B) = \text{bi-co}(A)$.

Now let g be a bi-affine function on B; we will show that it cannot separate the point $(\frac{1}{4},\frac{1}{4})$ from A. Indeed, let α, β, γ, δ be the values of g at the points of A: $(0,0)$, $(\frac{1}{2},0)$, $(0,\frac{1}{2})$, $(1,1)$ (respectively). Without loss of generality, assume

$g(\frac{1}{4},\frac{1}{4})=0$; thus, α, β, γ, δ are all negative. Using repeatedly the fact that g is bi-affine, we obtain:

$$g(\tfrac{1}{4},0)=\tfrac{1}{2}g(0,0)+\tfrac{1}{2}g(\tfrac{1}{2},0)=\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta,$$

$$g(\tfrac{1}{4},\tfrac{1}{2})=2g(\tfrac{1}{4},\tfrac{1}{4})-g(\tfrac{1}{4},0)=-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta,$$

$$g(\tfrac{1}{2},\tfrac{1}{2})=2g(\tfrac{1}{4},\tfrac{1}{2})-g(0,\tfrac{1}{2})=-\alpha-\beta-\gamma,$$

$$g(\tfrac{1}{2},1)=2g(\tfrac{1}{2},\tfrac{1}{2})-g(\tfrac{1}{2},0)=-2\alpha-3\beta-2\gamma,$$

$$g(0,1)=2g(0,\tfrac{1}{2})-g(0,0)=2\gamma-\alpha,$$

$$g(1,1)=2g(\tfrac{1}{2},1)-g(0,1)=-3\alpha-6\beta-6\gamma.$$

But $g(1,1)=\delta$, thus $-3\alpha-6\beta-6\gamma=\delta$, which is impossible since $\alpha,\beta,\gamma,\delta<0$. ∎

Finally, we show that the separation does depend on the domain of definition B (in the regular convex case, all the separation is obtained by affine functions, which can always be extended to the whole space; this is so for neither convex nor bi-convex functions).

EXAMPLE 3.5. Let $\mathscr{X}=\mathscr{Y}=[0,2]$, $A=\{(x,y)\mid 1<x,y<2 \text{ or } x=y=1\}$ (i.e., A is an open square together with one of its corners). Let $B=\mathscr{X}\times\mathscr{Y}$; then we claim that the points $(x,1)$ and $(1,y)$, for $1<x,y<2$, belong to $\operatorname{ns}(B)$. Indeed, let f be a bi-convex function on B, then

$$f(x,1)\leqq\frac{\varepsilon}{1+\varepsilon}f(x,0)+\frac{1}{1+\varepsilon}f(x,1+\varepsilon)$$

for every $0<\varepsilon<1$. Since $(x,1+\varepsilon)\in A$ for $1<x<2$ and $0<\varepsilon<1$, we obtain when $\varepsilon\to 0$ that $f(x,1)\leqq\sup f(A)$, thus $(x,1)\in\operatorname{ns}(B)$. Similarly for $(1,y)$.

Now let $B=[1,2)\times[1,2)$; then the following bi-convex function separates all points $(x,1)$ and $(1,y)$ (for $1<x,y<2$:

$$f(x,y)=\begin{cases} x-1, & y=1,\\ y-1, & x=1,\\ 0, & \text{otherwise.}\end{cases}$$ ∎

What are the continuity properties of bi-convex functions? As we shall now see, they parallel those of convex functions (cf. [5]). A real function f defined on a set B is *lower-semi-continuous* at a point $\bar{z}\in B$ if

$$\liminf_{z\to\bar{z}} f(z)=f(\bar{z})$$

(or, equivalently, if $\liminf_{n\to\infty} f(z_n) \geqq f(\bar{z})$ for every sequence $\{z_n\}_{n=1}^{\infty} \subset B$ such that $z_n \to \bar{z}$). It is *upper-semi-continuous* at $\bar{z}$ if

$$\limsup_{z\to\bar{z}} f(z) = f(\bar{z}),$$

and it is *continuous* at $\bar{z}$ if it is both lower- and upper-semi-continuous there. The following results should be compared with Theorems 7.4 and 10.2 in [5].

Let $B \subset \mathscr{X} \times \mathscr{Y}$ and let $z = (x, y) \in B$. The point z is *bi-relatively interior* to B if z is interior to B relative to $\mathrm{aff}(\mathrm{proj}_{\mathscr{X}} B) \times \mathrm{aff}(\mathrm{proj}_{\mathscr{Y}} B)$, where the affine space generated by a set C is denoted $\mathrm{aff}(C)$. For example, let $\mathscr{X} = \mathscr{Y} = [0,1]$ and let $B = \{(t,t) \mid 0 < t < 1\}$, then every point of B is a relatively interior point, but none is bi-relatively interior. Note also that on this set B, *any* function f is bi-convex!

PROPOSITION 3.6. *Let f be a bi-convex function on a bi-convex set B, and let $\bar{z} = (\bar{x}, \bar{y})$ be a bi-relatively interior point of B. Then f is lower-semi-continuous at $\bar{z}$.*

PROOF. Without loss of generality, assume $\bar{z}$ is actually interior to B. Let U be a closed cube around $\bar{x}$, and V a closed cube around $\bar{y}$, such that $U \times V \subset B$. Let $z = (x, y) \in U \times V$; express it as a bi-convex combination of the vertices of $U \times V$, say $z = \sum_{i=1}^{I} \alpha_i z_i$; then

$$f(z) \leqq \sum_{i=1}^{I} \alpha_i f(z_i),$$

which implies that f is bounded from above on $U \times V$ (by $\max\{f(z) \mid z \text{ vertex of } U \times V\}$).

Now let $(x, y) \in U \times V$; continue the straight line (in U) through x and $\bar{x}$, past $\bar{x}$, until it intersects the boundary of U at a point x'; define y' similarly. Then $\bar{x} = \lambda x + \lambda' x'$ and $\bar{y} = \mu y + \mu' y'$, where λ, λ', μ, $\mu' \geqq 0$, $\lambda + \lambda' = \mu + \mu' = 1$. Since f is a bi-convex function,

$$f(\bar{x}, \bar{y}) \leqq \lambda f(x, \bar{y}) + \lambda' f(x', \bar{y}) \leqq \lambda\mu f(x, y) + \lambda\mu' f(x, y') + \lambda' f(x', \bar{y}).$$

As $(x, y) \to (\bar{x}, \bar{y})$, we have $\lambda' \to 0$ and $\mu' \to 0$ (the boundaries of U and V are at a positive distance from $\bar{x}$ and $\bar{y}$, respectively). Together with the boundedness from above of f on $U \times V$, this implies that only the first term matters, thus

$$f(\bar{x}, \bar{y}) \leqq \liminf_{(x,y)\to(\bar{x},\bar{y})} f(x, y). \qquad \blacksquare$$

Again, let $B \subset \mathscr{X} \times \mathscr{Y}$ and $z = (x, y) \in B$. We say that B is *locally bi-simplicial at* z if there exist a neighborhood U of x in $\mathscr{X}$, a neighborhood V of y in $\mathscr{Y}$, a collection of simplices $S_1, S_2, \ldots, S_n$ in $\mathscr{X}$ and a collection of simplices $T_1, T_2, \ldots, T_m$ in $\mathscr{Y}$, such that (putting $S = \bigcup_{i=1}^n S_i$ and $T = \bigcup_{j=1}^m T_j$), $S \times T \subset B$ and $(U \times V) \cap B = (U \times V) \cap (S \times T)$ (compare with [5, p. 84]). Examples of sets that are locally bi-simplicial at all their points are sets $B = C \times D$, where $C \subset \mathscr{X}$ and $D \subset \mathscr{Y}$ are (relatively) open convex sets, or polyhedral sets. If we consider again $\mathscr{X} = \mathscr{Y} = [0, 1]$ and $B = \{(t, t) \mid 0 < t < 1\}$, then B is locally bi-simplicial at none of its points (although it is locally simplicial at all of them).

PROPOSITION 3.7. *Let f be a bi-convex function on a bi-convex set B, and let $\bar{z} = (\bar{x}, \bar{y}) \in B$. If B is locally bi-simplicial at $\bar{z}$, then f is upper-semi-continuous at $\bar{z}$.*

PROOF. Without loss of generality, assume each S_i has $\bar{x}$ as one of its vertices, and each T_j has $\bar{y}$ as one of its vertices (if this is not so, partition the corresponding simplex into smaller ones with this property). It suffices to show that f is upper-semi-continuous on each $S_i \times T_j$. Let $x_0 = \bar{x}, x_1, \ldots, x_p$ be the vertices of S_i, and $y_0 = \bar{y}, y_1, \ldots, y_q$ the vertices of T_j; then each $x \in S_i$ and each $y \in T_j$ can be expressed as $x = \sum_{r=0}^p \lambda_r x_r$ and $y = \sum_{s=0}^q \mu_s y_s$, with $\lambda_r, \mu_s \geqq 0$ and $\sum_{r=0}^p \lambda_r = \sum_{s=0}^q \mu_s = 1$. Hence

$$f(x, y) \leqq \sum_{r=0}^{p} \lambda_r f(x_r, y) \leqq \sum_{r=0}^{p} \sum_{s=0}^{q} \lambda_r \mu_s f(x_r, y_s).$$

As $(x, y) \to (\bar{x}, \bar{y})$, we have $\lambda_0 \to 1$ and $\mu_0 \to 1$, thus $\lambda_r \to 0$ and $\mu_s \to 0$ for all $r \neq 0$ and $s \neq 0$, implying that $\limsup f(x, y) \leqq f(\bar{x}, \bar{y})$. ∎

COROLLARY 3.8. *Let f be a bi-convex function on a bi-convex set B. Then f is continuous at all its bi-relatively interior points.*

PROOF. If z is a bi-relatively interior point of B, then B is locally bi-simplicial at z. ∎

We now complete the analysis of Examples 3.3 and 3.5. Consider first Example 3.3; we wish to show that $\mathrm{ns}(B)$ does not contain any points outside C. Indeed, the function

$$f(x, y) = [x - \tfrac{2}{3}]_+ [y - \tfrac{1}{3}]_+$$

(where $[\lambda]_+ \equiv \operatorname{Max}\{\lambda, 0\}$ for real λ), separates from A all points in the positive orthant with origin at w_4. In a similar way the other three orthants (with origins w_1, w_2 and w_3) are also separated.

Functions of this type, i.e.

$$f(x,y)=[g(x)]_+[h(y)]_+, \tag{3.9}$$

where g and h are affine functions on $\mathscr{X}$ and $\mathscr{Y}$, respectively, are often useful (for applications, see Section 7 in [3]). In Example 3.5, they suffice to show that

$$\mathrm{ns}(\mathscr{X}\times\mathscr{Y})=[1,2)\times[1,2),$$
$$\mathrm{ns}([1,2)\times[1,2))=A$$

(note that $A = A^*$, since A is a convex set). However, functions of this type do not separate everything that bi-convex functions do (see Example 3.4).

4. Main results

In this section we will obtain a characterization of A^* by separation properties. The main result is Theorem 4.7; see also Theorem 4.3 (these correspond to (1) and (2) at the end of the Introduction.)

In Section 3 we have defined, for every bi-convex set B that contains A, the set $\mathrm{ns}(B)$ of all points of B that cannot be separated from A by any bi-convex function. As we saw in Example 3.5, one may have to apply the operator "ns" repeatedly in order to obtain the desired set A^* (see also Example 5.5 and Remark 5.7).

Formally, one defines inductively $B_0=\mathscr{X}\times\mathscr{Y}$, $B_{\alpha+1}=\mathrm{ns}(B_\alpha)$ for every ordinal α, and $B_\alpha=\bigcap_{\beta<\alpha}B_\beta$ for every limit ordinal[7] α. Since $\mathrm{ns}(B)\subset B$ for every[8] B, and $B\subset B'$ implies $\mathrm{ns}(B)\subset\mathrm{ns}(B')$, one obtains a non-increasing sequence of sets $\{B_\alpha\}_\alpha$, with limit $C\equiv B_\gamma$ for some ordinal γ. (In the introduction, the limit set C was denoted bi-co$^{\#}(A)$.)

PROPOSITION 4.1. *The limit set C satisfies $C=\mathrm{ns}(C)$. Moreover, it is the largest such set, i.e., if $B=\mathrm{ns}(B)$ then $B\subset C$.*

PROOF. Since $C=B_\gamma$ is the limit of the above sequence, we have $B_{\gamma+1}=B_\gamma$, or $\mathrm{ns}(C)=C$. If $B=\mathrm{ns}(B)$, then $B\subset B_0=\mathscr{X}\times\mathscr{Y}$, and $B\subset B_\beta$ for all $\beta<\alpha$ implies $B=\mathrm{ns}(B)\subset\mathrm{ns}(B_\beta)$, thus $B\subset B_\alpha$; transfinite induction then gives $B\subset B_\gamma=C$. ∎

Does this set C coincide with A^*? Example 5.1 will show that this is not the case in general. What then is this set C? Consider Example 2.5: the points w_i

[7] Equivalently, define $B_\alpha=\bigcap_{\beta<\alpha}\mathrm{ns}(B_\beta)$ for *every* ordinal α. Note that one may take $B_0=\mathrm{co}(A)$.

[8] We will always assume throughout this section that B is a bi-convex set containing A.

(for $1 \leqq i \leqq 4$) belong to A^* but not to bi-co(A). Actually, w_i can be obtained by a bi-martingale which a.s. reaches A in finite time (i.e., one need not go to the limit Z_∞, since on almost every path $Z_n \in A$ for all n large enough; how large depends on the path).

Formally,[9] let $\{Z_n\}_{n=1}^{\infty}$ be a bi-martingale, $Z_n \to Z_\infty$ a.s., and let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be the corresponding sequence of (finite) fields; that is, $\mathcal{F}_n$ is the field generated by $(Z_1, Z_2, \dots, Z_n)$. Put $\mathcal{F}_\infty = \lim_{n\to\infty} \mathcal{F}_n$ (a σ-field), and denote by $\mathbf{N}$ the set of positive integers $\{1, 2, \dots\}$. A *stopping time* N is a random variable with values in $\mathbf{N}_\infty \equiv \mathbf{N} \cup \{\infty\}$, such that the event $\{N = n\}$ belongs to $\mathcal{F}_n$ for every $n \in \mathbf{N}_\infty$. Intuitively, this means that N depends only on the "past" — i.e., $Z_1, Z_2, \dots, Z_N$, but not on the "future". A stopping time N is *a.s. finite* if $P(N < \infty) = 1$; it is *a.s. bounded* if there exists $n_0 < \infty$ such that $P(N \leqq n_0) = 1$. Note that if we only consider values of Z_n that have positive probability, the finiteness of the fields $\mathcal{F}_n$ implies that "a.s. bounded" is the same as "everywhere bounded", which is the same as "everywhere finite" (by König's Lemma); this however differs from "a.s. finite" (see Fig. 2.3 for an example).

We now define $A^{\#}$ as the set of all $z \in \mathcal{X} \times \mathcal{Y}$ such that there exists a bi-martingale $\{Z_n\}_{n=1}^{\infty}$ with $Z_1 = z$, together with an a.s. finite stopping time N, such that $Z_N \in A$ (a.s.). Note that if we require the stopping time to be bounded, then bi-co(A) is obtained (see Remark 2.4), whereas A^* corresponds to the case that the stopping time need not be a.s. finite. In a similar way to Proposition 2.3, we have

PROPOSITION 4.2. *$A^{\#}$ is a bi-convex set, satisfying*

$$\text{bi-co}(A) \subset A^{\#} \subset A^*.$$

Example 2.5 shows that bi-co(A) may be a proper subset of $A^{\#}$; Example 5.1 will show that $A^{\#}$ may be a proper subset of A^*.

THEOREM 4.3. *The largest set C satisfying $C = \mathrm{ns}(C)$ is precisely $A^{\#}$.*

Thus, $A^{\#}$ is the largest set that contains A and such that no bounded bi-convex function defined on $A^{\#}$ can separate any of its points from A. We divide the proof into two parts.

PROPOSITION 4.4. $\mathrm{ns}(A^{\#}) = A^{\#}$.

PROOF. Let $z \in A^{\#}$, $\{Z_n\}_{n=1}^{\infty}$ a bi-martingale with $Z_1 = z$, and let N be an a.s. finite stopping time with $Z_N \in A$ (a.s.). For every bounded bi-convex function f

[9] References for the following are, e.g., [1, Ch. 9], [4, Ch. IV–V].

defined on $A^{\#}$, the sequence $\{f(Z_n)\}_{n=1}^{\infty}$ is a real bounded sub-martingale (i.e., $E[f(Z_{n+1}) \mid \mathcal{F}_n] \geqq f(Z_n)$ for every n; this follows from the fact that f is bi-convex and $\{Z_n\}$ is a bi-martingale[10]). Since N is an a.s. finite stopping time, we obtain $f(z) = f(Z_1) \leqq E(f(Z_N))$. But $Z_N \in A$ a.s., thus $f(Z_N) \leqq \sup f(A)$, hence $f(z) \leqq \sup f(A)$, which proves that $z \in \mathrm{ns}(A^{\#})$. ∎

PROPOSITION 4.5. *Let B satisfy $B = \mathrm{ns}(B)$. Then $B \subset A^{\#}$.*

PROOF. Define a real function $\varphi \equiv \varphi_B$ on B by $\varphi(z) = \inf P(Z_n \notin A$ for all $n \geqq 1)$ for every $z \in B$, where the infimum is taken over all bi-martingales $\{Z_n\}_{n=1}^{\infty}$ with $Z_1 = z$ and $Z_n \in B$ for all $n \geqq 1$. If is straightforward to check that φ is a non-negative bi-convex function on B, and moreover that $\varphi(a) = 0$ for all $a \in A$. Since $\mathrm{ns}(B) = B$ we cannot separate by φ, thus $\varphi(z) = 0$ for all $z \in B$.

Now $1 - \varphi(z) = \sup P(Z_n \in A$ for some $n \geqq 1) = \sup P(Z_N \in A)$, the supremum being taken over all bi-martingales $\{Z_n\}_{n=1}^{\infty}$ as above and over all a.s. finite stopping times N. Therefore it remains to prove that this supremum is achieved for each $z \in B$. This is a standard argument[11]; we will briefly sketch it here.

Choose $0 < \rho < 1$; for every $z \in B$, $\varphi(z) = 0$; hence there exists a bi-martingale $\{Z_n\}_{n=1}^{\infty}$ together with a stopping time N, such that $Z_1 = z$, $Z_n \in B$ for all n, $P(N < \infty) = 1$, and $P(Z_N \in A) > \rho$; since once A is reached, the bi-martingale can remain constant, we may replace N by an integer $m \equiv m(z)$ large enough, such that $P(Z_m \in A) > \rho$. (We will say that $\{Z_n\}$ and $m(z)$ "correspond" to z.)

Consider the bi-martingale $\{Z_n\}$ corresponding to z, and follow it up to step $m = m(z)$; from each point $z' = Z_m$ that does not belong to A (but does however belong to B), continue with the bi-martingale corresponding to z', for $m(z')$ more steps, and so on. The total probability that A is reached in finite time is then at least

$$\rho + (1-\rho)\rho + (1-\rho)^2\rho + \cdots,$$

which converges to 1. This completes the proof. ∎

PROOF OF THEOREM 4.3. Propositions 4.4 and 4.5 give the two inclusions $A^{\#} \subset C$ and $C \subset A^{\#}$, respectively. ∎

We have thus seen that separating by *all* bi-convex functions leads to $A^{\#}$, which is included in A^*. Now we will show that suitably restricting the family of functions used for separation leads to A^*.

[10] Note that $Z_n \in A^{\#}$ for all n, thus $f(Z_n)$ is well defined.

[11] E.g., it follows from Corollary 3.8.1 in [2].

Let B be a bi-convex set containing A. Let $\mathscr{C}(B) \equiv \mathscr{C}_A(B)$ be the set of all real functions on B that are bi-convex, bounded, and continuous at each point of A (continuity is not required on all B, but just on A). Note that functions of the type (3.9) belong to $\mathscr{C}(B)$ for any B. Let $\operatorname{nsc}(B)$ be the set of all $z \in B$ that are not separated from A by any $f \in \mathscr{C}(B)$; that is, such that $f(z) \leqq \sup f(A)$ for all $f \in \mathscr{C}(B)$. One immediately obtains

PROPOSITION 4.6. *For every B, the set* $\operatorname{nsc}(B)$ *is bi-convex, and* $\operatorname{ns}(B) \subset \operatorname{nsc}(B) \subset B$.

We now define the set D as the largest set such that $\operatorname{nsc}(D) = D$. As was the case for the set C, we obtain D as the limit of the sequence $\{B_\alpha\}_\alpha$, where $B_0 = \mathscr{X} \times \mathscr{Y}$ and $B_\alpha = \bigcap_{\beta<\alpha} \operatorname{nsc}(B_\beta)$ for all ordinals α. (In the introduction, the limit set D was denoted bi-co*(A).)

THEOREM 4.7. *Assume A is a closed set. Then the largest set D satisfying $D = \operatorname{nsc}(D)$ is precisely A^*.*

Thus, A^* is the largest set that contains A, and such that no bi-convex function defined on A^* and continuous on A, can separate any point in A^* from A.

PROOF. It will follow from Propositions 4.8 and 4.9.

PROPOSITION 4.8. $\operatorname{nsc}(A^*) = A^*$.

PROOF. Let $z \in A^*$, $\{Z_n\}_{n=1}^\infty$ a bi-martingale with $Z_1 = z$, $Z_n \to Z_\infty$, $P(Z_\infty \in A) = 1$, and let $f \in \mathscr{C}(A^*)$. Since $Z_n \in A^*$ for all n, we obtain a bounded (real) sub-martingale $\{f(Z_n)\}_{n=1}^\infty$; moreover, $Z_\infty \in A$ implies $f(Z_n) \to f(Z_\infty)$, therefore $f(z) = f(Z_1) \leqq E(f(Z_\infty)) \leqq \sup f(A)$. ■

PROPOSITION 4.9. *Assume A is a closed set, and let B satisfy $B = \operatorname{nsc}(B)$. Then $B \subset A^*$.*

PROOF. For every z, let $d(z, A)$ denote the distance of z from the set A; A being a closed set, $d(z, A) = 0$ if and only if $z \in A$. Define a real function $\psi \equiv \psi_B$ on B by $\psi(z) = \inf E[d(Z_\infty, A)]$ for every $z \in B$, where the infimum is taken over all bi-martingales $\{Z_n\}_{n=1}^\infty$ satisfying $Z_1 = z$, $Z_n \in B$ for all n, and $Z_n \to Z_\infty$ (a.s.). It is again easy to see that ψ is a bounded bi-convex function. One possible bi-martingale for z is the constant one (namely, $Z_n = z$ for all n); therefore $\psi(z) \leqq d(z, A)$, which shows that ψ vanishes and is continuous at every point of A. But $B = \operatorname{nsc}(B)$, hence ψ does not separate any point of B from A — thus ψ is identically zero on all B.

To complete the proof we will now show that the infimum in the definition of $\psi(z)$ is indeed achieved[12] for all $z \in B$. Since $Z_n \to Z_\infty$ implies $d(Z_n, A) \to d(Z_\infty, A)$, we have $E[d(Z_n, A)] \to E[d(Z_\infty, A)]$; therefore, for every $z \in B$ and every $\rho > 0$ there exists a bi-martingale $\{Z_n\}_{n=1}^\infty$ (with $Z_1 = z$ and $Z_n \in B$ for all n), and an integer m such that $E[d(Z_m, A)] < \rho$. After stage m, continue with bi-martingales corresponding to each $z' = Z_m$ and $\rho/2$, for $m' \equiv m'(z')$ more steps; follow then bi-martingales corresponding to $z'' = Z_{m+m'}$ and $\rho/3$, and so on. This construction yields a new bi-martingale $\{\bar{Z}_n\}_{n=1}^\infty$ with $\bar{Z}_1 = z$ and $\bar{Z}_n \in B$ for all n; let $\bar{Z}_\infty$ be its a.s. limit. We also obtain an increasing sequence $\{N_k\}_{k=1}^\infty$ of finite stopping times ($N_1 = m, N_2 = m + m', \ldots$) such that $E[d(\bar{Z}_{N_k}, A)] < \rho/k$ for all $k \geqq 1$. Therefore $E[d(\bar{Z}_\infty, A)] = 0$. ■

REMARK 4.10. It can be easily checked in the proof of Proposition 4.8 that it suffices for f to be just upper-semi-continuous rather than continuous at each point of A. Therefore Theorem 4.7 remains true if one allows separation by this type of bounded bi-convex functions too. In what regards checking upper-semi-continuity, recall Proposition 3.7.

REMARK 4.11. The finiteness of the fields $\mathcal{F}_n$ does not play any role in the proofs of Propositions 4.4 and 4.8. Together with Theorems 4.3 and 4.7, this implies that neither $A^\#$ nor A^* will change if this finiteness condition is dropped from the definition of a bi-martingale.

5. Examples

This section is devoted to three examples, settling (in the negative) some questions regarding A^*:

1. Is $A^\#$ equal to A^*?
2. Is $(A^*)^*$ equal to A^*?
3. If A is a closed set, is A^* closed too?

Since the motivation for the study of A^* came from game theory (see [3]), we will take in all three examples the set A to be compact and piecewise algebraic (i.e., a finite union of sets defined by algebraic functions). It is thus conjectured that the same phenomena appear in the game theoretic context as well.

All three examples use an idea similar to Example 2.2; however, by making each of the two spaces $\mathcal{X}$ and $\mathcal{Y}$ two-dimensional, one can obtain a kind of a "rotating staircase", which eliminates unwanted interaction between the various "steps". To get a geometric picture, imagine in Fig. 2.1 that $\mathcal{X}$ becomes

[12] Again, one may apply Corollary 3.8.1 in [2].

two-dimensional — a plane perpendicular to the page — whereas $\mathscr{Y}$ remains one-dimensional; "rotate" slightly each of the "steps" $w_2z_3, w_4z_5, \ldots$ in the $\mathscr{X}$-plane, around $w_3, w_5, \ldots$ (respectively).

EXAMPLE 5.1. Let $\mathscr{X} = \mathscr{Y} = [0,1]^2 \subset \mathbf{R}^2$. Let $T = [0, 0.2]$; for every $t \in T$, let[13]

$$b_t = (1, 3t - 2t^2; 2t, 4t^2), \qquad c_t = (t, t^2; 1, 3t - 2t^2),$$

$$d_t = (2t, 4t^2; 2t, 4t^2), \qquad e_t = (t, t^2; 2t, 4t^2).$$

Let $B = \{b_t\}_{t \in T}$, $C = \{c_t\}_{t \in T}$, $D = \{d_t\}_{t \in T \setminus \{0\}}$, $E = \{e_t\}_{t \in T \setminus \{0\}}$, and define $A = B \cup C \cup \{O\}$, where $O = (0,0;0,0)$.

PROPOSITION 5.2. (1) $D \cup E \subset A^*$.
(2) $(D \cup E) \cap A^{\#} = \varnothing$.

PROOF. (1) For every $t \in T$, $t \neq 0$, we have

$$d_t = \frac{t}{1-t} b_t + \frac{1-2t}{1-t} e_t \qquad (y \text{ constant}),$$

$$e_t = \frac{t}{1-t} c_t + \frac{1-2t}{1-t} d_{t/2} \qquad (x \text{ constant}).$$

We thus obtain a bi-martingale, represented in tree form in Fig. 5.1. As $t \to 0$, both d_t and e_t converge to $O \in A$; therefore the above bi-martingale converges

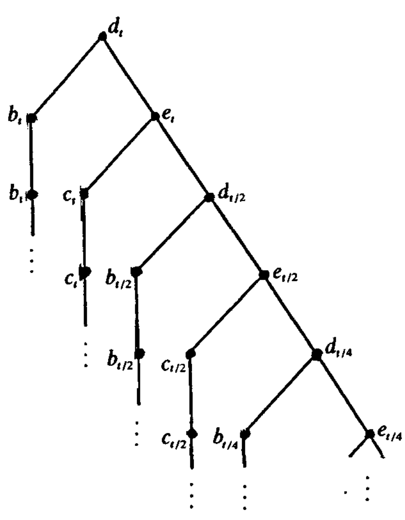

Fig. 5.1.

[13] We write a point z as $z = (x^{(1)}, x^{(2)}; y^{(1)}, y^{(2)})$, where $x = (x^{(1)}, x^{(2)}) \in \mathscr{X}$ and $y = (y^{(1)}, y^{(2)}) \in \mathscr{Y}$.

to A with probability one (recall that all b_t and c_t belong to A). Similarly for e_t, showing that d_t and e_t belong to A^* for all $t \in T \setminus \{0\}$.

(2) Let $\{Z_n\}_{n=1}^{\infty}$ be a bi-martingale in $\mathscr{X} \times \mathscr{Y}$ and let N be an a.s. finite stopping time, with $Z_1 = d_t$ (for some $t \in T$, $t \neq 0$) and $Z_N \in A$ (a.s.). Define now the sets

$$F_+ = \{z \in \mathscr{X} \times \mathscr{Y} \mid x^{(2)} > 0 \text{ and } y^{(2)} > 0\}, \qquad F_0 = \{z \in \mathscr{X} \times \mathscr{Y} \mid x^{(2)} = y^{(2)} = 0\},$$

and $F = F_+ \cup F_0$. Since F is a convex set and $F \supset A$, we obtain $F \supset A^{\#}$. Therefore $Z_n \in F$ for all n. Moreover, $Z_n \in F_+$ implies $Z_{n+1} \in F_+$, since $x^{(2)}$ and $y^{(2)}$ cannot both change. But $Z_1 = d_t \in F_+$, therefore $Z_n \in F_+$ for all n, hence $Z_N \in F_+$, which implies that $d_t \in (A \cap F_+)^{\#}$. Now $z \in A \cap F_+$ implies $z = b_t$ or $z = c_t$ for some $0 < t \leqq 0.2$, hence $x^{(1)} + y^{(1)} \geqq 1$. From this it follows that $x^{(1)} + y^{(1)} \geqq 1$ for every $z \in \mathrm{co}(A \cap F_+)$, hence for every $z \in (A \cap F_+)^{\#}$, but d_t does not satisfy this inequality. Similarly for e_t. ■

It is instructive to compute $\varphi \equiv \varphi_B$ for $B = \mathscr{X} \times \mathscr{Y}$ in this example (see the proof of Proposition 4.5). By considering the bi-martingale constructed in the proof of (1) above, we have (put $t_n \equiv t/2^n$):

$$\varphi(d_t) = \prod_{n=0}^{\infty} \left(\frac{1-2t_n}{1-t_n}\right)^2 = (1-2t)^2,$$

and

$$\varphi(e_t) = \frac{1-2t}{1-t} \prod_{n=1}^{\infty} \left(\frac{1-2t_n}{1-t_n}\right)^2 = (1-2t)(1-t).$$

As $t \to 0$, both $d_t \to O$ and $e_t \to O$, but $\varphi(d_t) \to 1$ and $\varphi(e_t) \to 1$; since $\varphi(O) = 0$, φ is indeed *not* continuous at O (which belongs to A).

The next example shows that the $*$ operator is not idempotent; namely, in general $(A^*)^* \supsetneqq A^*$. Actually, we will even show that $(A_2)^* \supsetneqq A^*$, where A_2 is the set of all bi-convex combinations of the elements of A (see Section 2).

Note, however, that $(A^{\#})^{\#} = A^{\#}$ (if N_1 and $N_2 \equiv N_2(\omega_{N_1})$ are a.s. finite stopping times, then so is $N_1 + N_2$). Thus, Example 5.3 provides a further instance of the "$*$" operator being different from the "$\#$" one (indeed: we must have either $A^* \neq A^{\#}$, or $A^* = A^{\#} \equiv B$ and then $B^* \neq B^{\#}$).

EXAMPLE 5.3. Let $\mathscr{X} = \mathscr{Y} = [-1,1]^2 \subset \mathbf{R}^2$. Let $T = [0, 0.2]$, and define the sets B, C, D and E as in Example 5.1. Let $g = (-1,0;0,0)$, and put $A = B \cup C \cup \{g\}$.

PROPOSITION 5.4. (1) $O \in A_2$ (*where* $O = (0,0;0,0)$).
(2) $D \cup E \subset (A_2)^*$.
(3) $(D \cup E) \cap A^* = \varnothing$.

PROOF. (1) $O = \frac{1}{2}g + \frac{1}{2}b_0$ (y is constant).

(2) Follows immediately from Proposition 5.2(1) and (1) above.

(3) Define the sets F_+, F_0 and F as in the proof of Proposition 5.2(2). Since $A \subset F$ and F is a convex set, we obtain $A^* \subset F$.

Let $z \in A^* \cap F_+$; we will show that $z \notin D \cup E$. Let $\{Z_n\}_{n=1}^{\infty}$ be a bi-martingale, $Z_1 = z$, $Z_n \to Z_\infty$, $Z_\infty \in A$ (a.s.). As in the previous Proposition, we again obtain $Z_n \in A^* \cap F_+$ for all n.

Consider the function

$$f(z) = f(x^{(1)}, x^{(2)}; y^{(1)}, y^{(2)}) = [-x^{(1)}]_+[y^{(2)}]_+$$

(see (3.9)). It is a bi-convex, bounded and continuous function. It vanishes on A (since $x^{(1)} < 0$ only at g, where $y^{(2)} = 0$), thus it must vanish on A^* (by Proposition 4.8). Therefore $Z_n \in A^* \cap F_+$ implies $0 \leqq X_n^{(1)}$ (= the first $\mathscr{X}$ coordinate of Z_n) hence $0 \leqq X_\infty^{(1)}$. Hence Z_∞ cannot equal g, and we have $Z_\infty \in B \cup C$ (a.s.) and $z \in (B \cup C)^*$.

Finally, $x^{(1)} + y^{(1)} \geqq 1$ on $B \cup C$, thus on $(B \cup C)^*$; but this is not so on $D \cup E$, completing the proof that $(D \cup E) \cap A^* = \emptyset$. ∎

The last example is concerned with topological properties of A^* and the other sets we dealt with: bi-co(A) and $A^\#$. If A is a closed set, so will be each of the sets A_n for $n \geqq 2$ (see Section 2). However, it may well be the case that none of bi-co(A), $A^\#$ and A^* are closed.

EXAMPLE 5.5. Let $\mathscr{X} = \mathscr{Y} = [-1,1]^2 \subset \mathbf{R}^2$. Let $T = [0, 0.1]$, $T' = [0.1, 0.2]$, and define for every $t \in T \cup T' = [0, 0.2]$

$$b_t = (-1, -3t - 2t^2; t, t^2), \qquad c_t = (2t, 4t^2; -1, -3t - 2t^2),$$

$$d_t = (t, t^2; t, t^2), \qquad e_t = (2t, 4t^2; t, t^2).$$

Let $B = \{b_t\}_{t \in T}$, $C = \{c_t\}_{t \in T}$, $D = \{d_t\}_{t \in T \setminus \{0\}}$, $D' = \{d_t\}_{t \in T'}$, $E = \{e_t\}_{t \in T \setminus \{0\}}$, and put $A = B \cup C \cup D'$.

PROPOSITION 5.6. (1) $D \cup E \subset$ bi-co(A).

(2) $O = (0,0;0,0) \notin A^*$.

Since $d_t, e_t \to O$ as $t \to 0$, the point O belongs to the *closure* of each one of the sets bi-co(A), $A^\#$ and A^*, but does not belong to any one of these sets.

PROOF. (1) For every $t \in T \setminus \{0\}$, d_t is a bi-convex combination of b_t and e_t (with y constant), and e_t is a bi-convex combination of c_t and d_{2t} (with x constant). Therefore bi-co(A) contains e_t for all $0.1/2 \leqq t \leqq 0.2/2$, hence d_t for all those t, hence e_t for all $0.1/4 \leqq t \leqq 0.2/4$, and so on. Thus d_t, $e_t \in$ bi-co(A) for all $0 < t \leqq 0.1$.

(2) Let $F = \{z \in \mathscr{X} \times \mathscr{Y} \mid x^{(2)} \leqq 0 \text{ and } y^{(2)} \leqq 0\}$ and $F_0 = \{z \in \mathscr{X} \times \mathscr{Y} \mid x^{(2)} = y^{(2)} = 0\}$. We claim that $A^* \cap F = A^* \cap F_0$. Indeed, for every $u > 0$ consider the function $[-3u - 2u^2 - x^{(2)}]_+[u^2 - y^{(2)}]_+$. It vanishes on A (since only b_t, for $t > u$, satisfies $x^{(2)} < -3u - 2u^2$; but then $u^2 - y^{(2)} = u^2 - t^2 < 0$). Therefore it vanishes on A^*. Let $z \in A^*$ with $x^{(2)} < 0$; then $x^{(2)} < -3u - 2u^2$ for some $u > 0$ (small enough), hence $y^{(2)} \geqq u^2 > 0$. In a similar way, $y^{(2)} < 0$ implies $x^{(2)} > 0$, thus $z \in A^* \cap F$ only when $x^{(2)} = y^{(2)} = 0$, or $z \in A^* \cap F_0$.

Consider now a bi-martingale $\{Z_n\}_{n=1}^{\infty}$ with $Z_1 = O$, $Z_n \to Z_\infty$, $Z_\infty \in A$ (a.s.). We claim that $Z_n \in A^* \cap F_0$ implies $Z_{n+1} \in A^* \cap F_0$. Indeed, assume without loss of generality that $X_{n+1} = X_n$. Then $X_{n+1}^{(2)} = X_n^{(2)} = 0$ and $E(Y_{n+1}^{(2)} \mid \mathscr{F}_n) = Y_n^{(2)} = 0$. If $Y_{n+1}^{(2)} \leqq 0$, then $Z_{n+1} \in A^* \cap F = A^* \cap F_0$, or $Y_{n+1}^{(2)} = 0$; thus $Y_{n+1}^{(2)} = 0$ throughout. Now $Z_1 = O \in A^* \cap F_0$, therefore $Z_n \in A^* \cap F_0$ for all n, implying that $Z_\infty \in F_0$. But $A \cap F_0 = \{b_0, c_0\}$, thus $O \in \{b_0, c_0\}^*$, which is clearly impossible (both b_0 and c_0 satisfy $x^{(1)} + y^{(1)} = 1$). ∎

Remark 5.7. In Example 5.5, the point O is a bi-relatively interior point of $\mathscr{X} \times \mathscr{Y}$, therefore any bi-convex function is continuous there (recall Corollary 3.8). Therefore O belongs to both ns$(\mathscr{X} \times \mathscr{Y})$ and nsc$(\mathscr{X} \times \mathscr{Y})$. This shows that even if A is a closed set — as in Example 5.5 — one may have to apply the operators ns and nsc more than once in order to obtain $A^{\#}$ and A^*, respectively (in Example 3.5, the set A was not closed).

References

1. K. L. Chung, *A Course in Probability Theory*, Academic Press, New York, 1974.
2. L. E. Dubins and L. J. Savage, *Inequalities for Stochastic Processes: How to Gamble if You Must*, Dover, New York, 1976.
3. S. Hart, *Nonzero-sum two-person repeated games with incomplete information*, Mathematics of Operations Research **10** (1985), 117–153.
4. P. A. Meyer, *Probability and Potentials*, Blaisdell Publishing Co., 1966.
5. T. R. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.